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A METHOD OF INTEGRATING THE EQUATIONS OF MOTION OF NON-HOLONOMIC SYSTEMS WITH HIGHER-ORDER CONSTRAINTS*

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An explicit form of the equations of motion of non-holonomic systems with higher-order constraints is studied and a field method /1/ is used to integrate them. Various methods of integrating the equations of motion of non-holonomic systems were discussed earlier in /2-7/. The generalization of the Hamilton-Jacobi method to non-holonomic systems has strict limitations /5, 6/. The field method /1/ was extended in /4/ to cover non-holonomic systems with first-order constraints.

1. Let the position of a mechanical system be described by the generalized coordinates q_s $(s = 1, \ldots, n)$ and m-th order constraints of the type

$$q_{\varepsilon+\beta}^{(m)} = \varphi_\beta (q_s, q_s^{\cdot}, \ldots, q_s^{(m-1)}, q_\sigma^{(m)}, t) \quad (\varepsilon = n - g) \tag{1.1}$$

Here and henceforth $\beta = 1, \ldots, g;\ s, h = 1, \ldots, n,\ \sigma, \nu = 1, \ldots, \varepsilon;\ m = 0, 1, 2, \ldots$
The equations of motion of the system are obtained in the form /9/

$$\frac{d}{dt}\frac{\partial T}{\partial q_\sigma^{\cdot}} - \frac{\partial T}{\partial q_\sigma^{\cdot}} - Q_\sigma + \sum_\beta \left(\frac{d}{dt}\frac{\partial T}{\partial q_{\varepsilon+\beta}^{\cdot}} - \frac{\partial T}{\partial q_{\varepsilon+\beta}} - Q_{\varepsilon+\beta}\right)\frac{\partial \varphi_\beta}{\partial q_\sigma^{(m)}} = 0 \tag{1.2}$$

Let us write

$$f_s (q_k, q_k^{\cdot}, q_k^{\cdot\cdot}, t) \equiv \frac{d}{dt}\frac{\partial T}{\partial q_s^{\cdot}} - \frac{\partial T}{\partial q_s} - Q_s \tag{1.3}$$

$$a_{\beta\sigma} (q_s, q_s^{\cdot}, \ldots, q_s^{(m-1)}, q_\nu^{(m)}, t) \equiv \partial \varphi_\beta / \partial q_\sigma^{(m)}$$

Eqs.(1.2) will now become

$$\sum_\beta f_{\varepsilon+\beta} a_{\beta\sigma} + f_\sigma = 0 \tag{1.4}$$

Let us consider the explicit form of Eqs.(1.1) and (1.4).

When $m = 0$, Eqs.(1.1) are holonomic and the order of Eqs.(1.4) is 2ε. Thus we have a holonomic system with redundant coordinates. Differentiating Eqs.(1.1) twice with respect to t, we obtain

$$q_{\varepsilon+\beta}^{\cdot\cdot} = \sum_\sigma \frac{\partial \varphi_\beta}{\partial q_\sigma} q_\sigma^{\cdot\cdot} + \sum_\sigma \sum_\nu \frac{\partial^2 \varphi_\beta}{\partial q_\sigma \partial q_\nu} q_\sigma^{\cdot} q_\nu^{\cdot} + 2 \sum_\sigma \frac{\partial^2 \varphi_\beta}{\partial q_\sigma \partial t} q_\sigma^{\cdot} + \frac{\partial^2 \varphi_\beta}{\partial t^2} \tag{1.5}$$

Eqs.(1.4) are linear in $q_s^{\cdot\cdot}$, and Eqs.(1.4), (1.5) can be solved in generalized coordinates

$$q_s^{\cdot\cdot} = h_s (q_k, q_k^{\cdot}, t) \tag{1.6}$$

The order of Eqs.(1.1) and (1.4) is 2ε, and that of Eq.(1.6) is $2n$. In order to obtain from Eqs.(1.6) the solution of Eqs.(1.1) and (1.4) for initial conditions $(q_s)_0$, $(q_s^{\cdot})_0$, we must impose the following constraints on $(q_s)_0$, $(\dot q_s)_0$:

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$$(q_{\varepsilon+\beta})_0 = \varphi_\beta((q_\sigma)_0, 0), \quad (\dot{q}_{\varepsilon+\beta})_0 = \sum_\sigma \left(\frac{\partial\varphi_\beta}{\partial q_\sigma}\right)_0 (q_\sigma^{\cdot})_0 + \left(\frac{\partial\varphi_\beta}{\partial t}\right)_0 \tag{1.7}$$

When $m = 1$, Eqs.(1.1) will represent non-holonomic first-order constraints, and we shall have the problems of non-holonomic first-order systems. Differentiating (1.1) with respect to t, we obtain

$$\ddot{q}_{\varepsilon+\beta} = \sum_{\sigma=1}^{\varepsilon} \frac{\partial\varphi_\beta}{\partial v_\sigma^{\cdot}} q_\sigma^{\cdot\cdot} + \sum_{s=1}^{n} \frac{\partial\varphi_\beta}{\partial q_s} q_s^{\cdot} + \frac{\partial\varphi_\beta}{\partial t} \tag{1.8}$$

Eqs.(1.4) and (1.8) yield the generalized accelerations $q_s^{\cdot\cdot}$ in the form (1.6). The order of Eqs.(1.1) and (1.4) is $q + 2\varepsilon$. In order to obtain from Eqs.(1.6) the solution of Eqs.(1.1) and (1.4) with initial conditions $(q_s)_0, (q_s^{\cdot})_0$, we can impose the following constraints on $(q_s)_0, (q_s^{\cdot})_0$:

$$(\dot{q}_{\varepsilon+\beta})_0 = \varphi_\beta((q_s)_0, (q_\sigma^{\cdot})_0, 0) \tag{1.9}$$

When $m = 2$, Eqs.(1.1) will represent non-holonomic second-order constraints. If $\partial\varphi_\beta/\partial q_\sigma^{(m)}$ contain no $q^{\cdot\cdot}$, then Eqs.(1.4) will be linear in $q_s^{\cdot\cdot}$, otherwise they will be non-linear. We assume that the system of Eqs.(1.1) and (1.4) has a solution in $q_s^{\cdot\cdot}$, and in this case we can write them in the form (1.6).

When $m > 2$, the order of Eqs.(1.2) will range from 2ε to $m\varepsilon$, depending on the form of $\partial\varphi_\beta/\partial q_\sigma^{(m)}$. If the higher-order differential of the generalized coordinates in t is $l\ (0 \leqslant l \leqslant m)$, then on differentiating Eqs.(1.2) $m - 2$ times with respect to t $(l \leqslant 2)$ or $m - l$ times with respect to $t\ (l \geqslant 2)$, their order will become $m\varepsilon$. Let us combine these equations with (1.1), and assume that the system can be solved for $q_s^{(m)}$.

$$q_s^{(m)} = h_s(q_k, q_k^{\cdot}, \ldots, q_k^{(m-1)}, t) \quad (m > 2) \tag{1.10}$$

Then, in order to obtain from Eq.(1.10) a solution of Eqs.(1.1) and (1.4) with initial conditions $(q_s)_0, (q_s^{\cdot})_0$, we can impose constraints on Eq.(1.2). For example, when $l = 1, m = 4$, we have the following constraints:

$$\sum_\beta (f_{\varepsilon+\beta})_0 (a_{\beta\sigma})_0 + (f_\sigma)_0 = 0 \quad \sum_\sigma [(\dot{f}_{\varepsilon+\beta})_0 (a_{\beta\sigma})_0 + (f_{\varepsilon+\beta})_0 (\dot{a}_{\beta\sigma})_0] + (f_\sigma^{\cdot})_0 = 0 \tag{1.11}$$

When $l = 3, m = 4$, we have the constraints corresponding to the first equation of (1.11) only, etc.

Thus we can write the equations of motion for the general non-holonomic m-th order systems in the following explicit form:

$$q_s^{(m)} = h_s(q_k, \dot{q}_k, \ldots, q_k^{(m-1)}, t) \quad (m \geqslant 2) \tag{1.12}$$

Let us now transform the equations of motion (1.12) to a system of first-order equations. Let

$$x_s = q_s, \quad x_{n+s} = q_s^{\cdot}, \ldots, x_{(m-1)n+s}\, q_s^{(m-1)}$$

Then Eqs.(1.12) will take the form of a standard system of equations

$$\begin{gathered} x_s^{\cdot} = x_{n+s},\ x_{n+s} = x_{2n+s}, \ldots, \dot{x}_{(m-2)n+s} = x_{(m-1)n+s} \\ \dot{x}_{(m-1)n+s} = h_s(x_k, x_{n+k}, \ldots, x_{(m-1)n+k}, t) \end{gathered} \tag{1.13}$$

2. Let us consider the generalization of the field method. Using the field method /1, 8/ we select a variable, e.g. x_1 as a function of time t and of the remaining variables x_A $(A = 2, \ldots, mn)$:

$$x_1 = u(t, x_A) \tag{2.1}$$

Differentiating Eq.(2.1) with respect to t and using Eqs.(1.13), we obtain

$$\begin{gathered} \frac{\partial u}{\partial t} + \sum_{a=2}^{n} \frac{\partial u}{\partial x_a} x_{n+a} + \sum_s \frac{\partial u}{\partial x_{n+s}} x_{2n+s} + \ldots + \sum_s \frac{\partial u}{\partial x_{(m-2)n+s}} x_{(m-1)n+s} + \\ \sum_s \frac{\partial u}{\partial x_{(m-1)n+s}} h_s - x_{n+1} = 0 \end{gathered} \tag{2.2}$$

We shall call the quasilinear Eq.(2.2) the fundamental partial differential equation.

If the complete solution of Eq.(2.2) has the form

$$x_1 = u\,(t, x_A, C_\alpha) \quad (A_2 = 2, \ldots, mn;\ \alpha = 1, \ldots, mn) \tag{2.3}$$

then substitution of expression (2.3) will yield an identity. Let us denote the initial values of the field variables by

$$x_\alpha\,(0) = x_{\alpha 0} \quad (\alpha = 1, \ldots, mn) \tag{2.4}$$

Substituting expression (2.4) into (2.3) and denoting one constant, e.g. C_1, by x_{α_0} and the remaining constants by C_A, we obtain

$$x_1 = u\,(t, x_A, x_{\alpha 0}, C_A) \tag{2.5}$$

It can be proved that /7/ when

$$\det\,(\partial^2 u/\partial C_A \partial x_\beta) \neq 0 \tag{2.6}$$

the solution of Eqs.(1.13) will be given, by the initial conditions, by relation (2.5) and we shall have $mn-1$ for $C_A\,\partial u/\partial C_A = 0\ (A = 2, \ldots, mn)$.

It should be noted that the order of the equations increases during the passage from (1.4) to (1.12). If the order increases by one, Eqs.(1.4) will become 2ε constraints for the initial conditions. If the order is increased by two, Eqs.(1.4) and their differentials with respect to t will become 4ε constraints for initial conditions, etc.

Using this method we can, in principle, integrate the equations of motion of non-holonomic m-th order systems. Using the field method we can choose the field variable so as to solve Eq.(2.2) easily. The main difficulty of this method lies in finding the solution of (2.2). However, having obtained one specific solution of this equation we can obtain the solution of the system from Eq.(2.5).

3. *Example 1.* Let us consider the motion of a point of unity mass, acted upon by the force $F_x = F_y = 0$, $F_z = k = \text{const}$, where the constaint equation has the form

$$z^{\cdot\cdot} = t x^{\cdot\cdot\cdot} + \sqrt{1+t^2}\, y^{\cdot\cdot\cdot} \tag{3.1}$$

Eqs.(1.2) yield

$$x^{\cdot\cdot} + t\,(z^{\cdot\cdot} - k) = 0, \quad y^{\cdot\cdot} + \sqrt{1+t^2}\,(z^{\cdot\cdot} - k) = 0 \tag{3.2}$$

Differentiating (3.2) with respect to t, adding the result to (3.1) and assuming that $x_1 = x$, $x_2 = y$, $x_3 = z$, we obtain

$$x_1^{\cdot} = x_4,\ x_2^{\cdot} = x_5,\ x_3^{\cdot} = x_6,\ x_4^{\cdot} = x_7,\ x_5^{\cdot} = x_8,\ x_6^{\cdot} = x_9 \tag{3.3}$$

$$x_7^{\cdot} = \frac{k - x_9}{1+t^2}, \quad x_8^{\cdot} = 0, \quad x_9^{\cdot} = \frac{(k - x_9)\,t}{1+t^2}$$

Let $x_1 = u\,(t, x_2, x_3, \ldots, x_9)$. The basic Eq.(2.2) yields

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x_2} x_5 + \frac{\partial u}{\partial x_3} x_6 + \frac{\partial u}{\partial x_4} x_7 + \frac{\partial u}{\partial x_5} x_8 + \frac{\partial u}{\partial x_6} x_9 + \frac{\partial u}{\partial x_7}\,\frac{k - x_9}{1+t^2} + \frac{\partial u}{\partial x_9}\,\frac{(k - x_9)\,t}{1+t^2} - x_4 = 0 \tag{3.4}$$

Let us assume that the complete solution of this equation has the form

$$x_1 = u = f_0\,(t) + \sum_{A=2}^{9} f_A\,(t)\,x_A \tag{3.5}$$

Substituting (3.5) into (3.4) and equating the free term and terms containing $x_2, x_3, \ldots, x_9$, we obtain

$$f_0^{\cdot} + f_7\frac{k}{1+t^2} + f_9\frac{kt}{1+t^2} = 0, \quad f_2^{\cdot} = 0, \quad f_3^{\cdot} = 0, \quad f_4^{\cdot} - 1 = 0$$

$$f_5^{\cdot} + f_2 = 0,\ f_6^{\cdot} + f_3 = 0,\ f_7^{\cdot} + f_4 = 0,\ f_8^{\cdot} + f_5 = 0$$

$$f_9^{\cdot} + f_6 - f_7\frac{1}{1+t^2} - f_9\frac{t}{1+t^2} = 0$$

Integrating, taking the initial conditions $C_0 = x_{10} - \sum_{A=2}^{9} C_A x_{A0}$ $(C_i = f_i\,(t_0))$ into account, and substituting into (3.4), we obtain

$$x_1 = u = x_{10} - \sum_{A=2}^{9} C_A x_{A0} + k\,(C_9 - C_3 - C_4) - k\,\{(C_9 - C_3 - C_4)\sqrt{1+t^2} + \tag{3.6}$$

$${}^1/_2\, C_3 t^2 - (C_6 + {}^1/_2)\,[\sqrt{1+t^2}\quad L\,(t) - t] + C_7 t\} + C_2 x_2 + C_3 x_3 + (C_4 + t)\,x_4 +$$

$$(C_5 - C_2t)\, x_5 + (C_6 - C_3t)\, x_6 + (C_7 - C_4t - {}^1/_2t^2)\, x_7 + (C_8 - C_5t + {}^1/_2\, C_2t^2)\, x_8 +$$
$$\{(C_9 - C_3 - C_4)\sqrt{1+t^2} + C_3(1+t^2) - C_6\sqrt{1+t^2}\, L(t) + C_7t + C_4 +$$
$${}^1/_2t - {}^1/_2\sqrt{1+t^2}\, L(t)\}\, x_9,\;\; L(t) = \ln(t + \sqrt{1+t^2})$$

The algebraic equations $\partial u/\partial C_A = 0\ (A = 2, \ldots, 9)$ yield

$$\begin{gathered} x_2 = x_{20} + x_{50}(t) + {}^1/_2x_{80}t^2,\; x_3 = x_{30} + x_{60}t + {}^1/_2kt^2 + (x_{90} - k)[tL(t) - M(t)] \\ x_4 = x_{40} + x_{70}t + (k - x_{90})M(t),\; x_5 = x_{50} + x_{80}t \\ x_6 = x_{60} + kt + (x_{90} - k)L(t),\; x_7 = x_{70} + (k - x_{90})t/\sqrt{1+t^2} \\ x_8 = x_{80},\; x_9 = k + (x_{90} - k)/\sqrt{1+t^2} \\ M(t) = \sqrt{1+t^2} - 1 \end{gathered} \tag{3.7}$$

Substituting expressions (3.7) into (3.6), we obtain

$$x_1 = x_{10} + x_{40}t + {}^1/_2x_{70}t^2 + {}^1/_2(k - x_{90})[L(t) + tM(t) - t] \tag{3.8}$$

Relations (3.7) and (3.8) represent the solution of Eqs.(3.4) for initial conditions $x_\alpha(0) = x_{\alpha 0}\ (\alpha = 1, \ldots, 9)$.

Finally we obtain the solution of the initial problem. The increase in the order on passing from (3.2) to (3.3) should be noted. Constraints (3.2) imposed on the initial accelerations yield

$$x_{70} = 0,\; x_{80} + (x_{90} - k) = 0 \tag{3.9}$$

Substituting relations (3.9) into (3.7) and (3.8), we obtain a solution containing seven constants corresponding to the order of the system (3.1), (3.2).

Example 2. (Appel's example). In Appel's example the Lagrange function and constraint equation have the form

$$L = {}^1/_2m(x^{\cdot 2} + y^{\cdot 2} + z^{\cdot 2}) - mgz,\;\; z^{\cdot} = ba^{-1}(x^{\cdot 2} + y^{\cdot 2})^{1/2} \tag{3.10}$$

Let $q_1 = x,\ q_2 = y,\ q_3 = z.$ Differentiating the last equation of (3.10) with respect to t, we obtain a second-order constraint

$$q_3^{\cdot\cdot} = ba^{-1}(q_1^{\cdot}q_1^{\cdot\cdot} + q_2^{\cdot}q_2^{\cdot\cdot})(q_1^{\cdot 2} + q_2^{\cdot 2})^{-1/2} \tag{3.11}$$

Eqs.(1.2) yield

$$mq_\alpha^{\cdot\cdot} + (mq_3^{\cdot\cdot} + mq)\, ba^{-1}q_\alpha^{\cdot}(q_1^{\cdot 2} + q_2^{2})^{-1/2} = 0,\quad \alpha = 1,2$$

from which we obtain, taking into account relations (3.10) and (3.11) and assuming that $x_1 = q_1,\ x_2 = q_2,\ x_3 = q_3,$

$$\begin{gathered} x_1^{\cdot} = x_4,\; x_2^{\cdot} = x_5,\; x_3^{\cdot} = x_6,\; x_4^{\cdot} = -Gx_4/x_6,\; x_5^{\cdot} = -Gx_5/x_6,\; x_6^{\cdot} = -G \\ (G = gb^2/(a^2 + b^2) \end{gathered} \tag{3.12}$$

Let

$$x_6 = u(t, x_1, x_2, \ldots, x_5)$$

Then the basis Eq.(2.2) will yield

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x_1}x_4 + \frac{\partial u}{\partial x_2}x_5 + \frac{\partial u}{\partial x_3}u + \frac{\partial u}{\partial x_4}\left(-G\,\frac{x_4}{u}\right) + \frac{\partial u}{\partial x_5}\left(-G\,\frac{x_5}{u}\right) + G = 0$$

Noting that

$$x_4 = Ax_6,\; x_5 = Bx_6\ (A = x_{40}/x_{60},\ B = x_{50}/x_{60})$$

we obtain

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x_1}Au + \frac{\partial u}{\partial x_2}Bu + \frac{\partial u}{\partial x_3}u - \frac{\partial u}{\partial x_4}GA - \frac{\partial u}{\partial x_5}GB + G = 0 \tag{3.13}$$

Let us write the solution in the form

$$x_6 = u = f_1 + f_2x_1 + f_3x_2 + f_4x_3 + f_5x_4 + f_6x_5\ (f_\alpha = f_\alpha(t),\ \alpha = 1, \ldots, 6)$$

and substitute it into Eq.(3.13). Equating the free term and terms containing $x_1, \ldots, x_5,$ we obtain

$$\begin{gathered} f_1^{\cdot} + (f_2A + f_3B + f_4)f_1 + (1 - f_5A - f_6B)G = 0 \\ f_a^{\cdot} + (f_2A + f_3B + f_4)f_a = 0\quad (a = 2, 3, 4, 5, 6) \end{gathered}$$

Integration yields

$$x_6 = u = [1 + (AC_2 + BC_3 + C_4)t]^{-1}\{C_1 + G[(AC_5 + BC_6 - 1)t - {}^1/_2(AC_2 + BC_3 + C_4)t^2] + C_2x_1 + C_3x_2 + C_4x_3 + C_5x_4 + C_6x_5\} \tag{3.14}$$

Let the initial conditions be

$$x_\alpha(0) = x_{\alpha 0} \quad (\alpha = 1, \ldots, 6) \tag{3.15}$$

Substituting (3.15) into (3.14), we obtain

$$C_1 = x_{60} - C_2x_{10} - C_3x_{20} - C_4x_{30} - C_5x_{40} - C_6x_{50} \tag{3.16}$$

and we follow this by eliminating C_1 from (3.14).

Algebraic Eqs. (2.5) yield

$$\partial u/\partial C_a = 0 \quad (a = 2, 3, 4, 5, 6)$$

Putting $C_a = 0$ and simplifying, we obtain

$$x_1 = x_{10} + x_{40}t - \frac{G}{2}\frac{x_{40}}{x_{60}}t^2, \quad x_2 = x_{20} + x_{50}t - \frac{G}{2}\frac{x_{50}}{x_{60}}t^2 \tag{3.17}$$
$$x_3 = x_{30} + x_{60}t - \frac{G}{2}t^2, \quad x_4 = x_{40} - G\frac{x_{40}}{x_{60}}t$$
$$x_5 = x_{50} - G\frac{x_{50}}{x_{60}}t$$

Substituting these expressions into (3.14) and taking into account Eqs.(3.16), we obtain

$$x_6 = x_{60} - Gt \tag{3.18}$$

Thus relations (3.17), (3.18) represent a solution of system (3.12) with initial conditions (3.15).

Finally we obtain the solution of Appel's problem. Its order is equal to five, and the order of system (3.12) is six. In order to obtain the solution of the problem we can impose a constraint on the constants in (3.17) and (3.18). It represents a restriction imposed on the non-holonomic constraint for the initial conditions, i.e.

$$x_{60} = ba^{-1}\sqrt{x_{40}^2 + x_{50}^2} \tag{3.19}$$

This implies that relations (3.17) and (3.19) represent a solution of the Appel's problem and contain five constants.

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